Elementary fixed points of the BRW smoothing transforms with infinite number of summands

Aleksander M. Iksanov*

Faculty of Cybernetics, Kiev Taras Shevchenko National University, 01033 Kiev, Ukraine

Abstract

The branching random walk (BRW) smoothing transform T is defined as $T : \mathrm{distr}(U_1) \mapsto \mathrm{distr}\left(\sum_{i=1}^{L} X_i U_i\right)$, where given realizations $\{X_i\}_{i=1}^{L}$ of a point process, $U_1, U_2, \ldots$ are conditionally independent identically distributed random variables, and $0 \le \mathrm{Prob}\{L = \infty\} \le 1$. Given $\alpha \in (0, 1]$, α-*elementary* fixed points are fixed points of T whose Laplace-Stieltjes transforms φ satisfy $\lim_{s \to +0} \frac{1 - \varphi(s)}{s^{\alpha}} = m$, where m is any given positive number. If $\alpha = 1$, these are the fixed points with finite mean. We show exactly when elementary fixed points exist. In this case these are the only fixed points of T and are unique up to a multiplicative constant. These results do not need any moment conditions. In particular, Biggins' martingale convergence theorem is proved in full generality. Essentially we apply recent results due to Lyons (1997) and Goldie and Maller (2000) as the key point of our approach is a close connection between fixed points with finite mean and perpetuities. As a by-product, we lift from our general results the solution to a Pitman-Yor problem. Finally, we study the tail behaviour of some fixed points with finite mean.

MSC: Primary: 60E10, 47H10, 60J80 ; Secondary: 60E07

Keywords: fixed points; smoothing transform; branching random walk; regular variation; perpetuity; Contraction Principle

*e-mail address: iksan@unicyb.kiev.ua

1 Introduction

Unless otherwise stated, all random variables (rvs) studied in the paper are assumed to be defined on a fixed probability space $(\Omega, \mathcal{F}, \mathbb{P})$. We also assume that this probability space is large enough to accomodate independent copies of rvs. All distributions mentioned below will be probability ones. Therefore the adjective will be usually dropped. The distribution of an rv $X = X(\omega)$, $\omega \in \Omega$ will be denoted by $\mathcal{L}(X)$, and the degenerate distribution at $x \geq 0$ (the delta measure) will be denoted by δ_x. Furthermore, $\mathcal{P}^+$ denotes the set of all Borel probability measures on the nonnegative half line $\mathbb{R}^+ = [0, \infty)$.

Let $Z(\cdot)$ be a point process on $[0, +\infty)$, i.e. a random, locally finite on $(0, +\infty)$, counting measure. It is assumed that realizations (points) $\{X_i\}_{i=1}^L$ of Z constitute a nonincreasing collection of L nonzero rvs, where L is an rv with $\mathbb{P}\{L = \infty\} \in [0, 1]$. We consider the ordered collection just for notational convenience, and this does not restrict generality. Note that $\mathbb{E}Z[0, t]$ and $\mathbb{E}Z(t, +\infty)$ may be infinite. Hence, the intensity measure χ of Z is defined for some positive finite A as follows $\int_A^{t+} \chi(dz) = \mathbb{E}Z(A, t]$, if $t \geq A$, and $\int_{t-}^A \chi(dz) = \mathbb{E}Z(t, A]$, otherwise. Note that in general χ is a σ-finite Borel measure.

Let us now recall what *the branching random walk (BRW)* is. Assume that an initial ancestor is placed at the origin of the real line and after one unit of time she gives birth to children who form the first generation. Their displacements from the origin are given by the point process $Z^{(1)}(B) := Z(e^{-B})$, where B is a Borel set and $e^{-B} = \{e^{-x} : x \in B\}$, with points $\{-\log X_i\}_{i=1}^L$. Each of these children also lives one unit of time and has offspring in a like manner, so that the positions of each family relative to the parent are given by an independent copy of the point process $Z^{(1)}$. All children born to individuals of the first generation forms the second generation with positions given by the point process $Z^{(2)}$ and so on. Thus $Z^{(n)}$ is the n-th generation point process. The discrete time process $Z^{(0)}(B) := 1_{\{\{0\} \in B\}}$ a.s., $Z^{(n)}$, $n = 1, 2, \ldots$, is called the BRW.

Let $\mathcal{F}_n$ be the σ-fields containing all information about the first n generations, $n = 1, 2, \ldots$. It is well-known that, when the mean number m of children born to a person satisfies $m \in (1, \infty]$, and $m(\gamma) := \mathbb{E} \int_{-\infty}^{\infty} e^{-\gamma t} Z^{(1)}(dt) \in (0, \infty)$, for some $\gamma \geq 0$,

$$W^{(n)}(\gamma) = (1/m^n(\gamma)) \int_{-\infty}^{\infty} e^{-\gamma t} Z^{(n)}(dt) \tag{1}$$

is a nonnegative martingale with respect to $\mathcal{F}_n$. For more information on the BRW and associated martingales see, for example, Biggins (1977), Biggins and Kyprianou (1997).

Let t_r be a rooted family tree associated with a point process $Z^{(1)}$. We say that (t_r, X) is a labelled tree if each individual (vertex) $\theta \in t_r \backslash \{0\}$ is assigned its displacement $X(\theta)$ from its parent. The BRW defines a probability measure μ on the set of labelled trees.

We address the problem of the existence and uniqueness of *special* distributions of the *nonnegative* rvs W satisfying the following distributional equality

$$W \stackrel{d}{=} \sum_{i=1}^{L} X_i W_i, \tag{2}$$

where $W_1, W_2, \ldots$ are, conditionally on $\{X_i\}_{i=1}^{L}$, independent copies of W. The equality (2) is equivalent to

$$\varphi(s) = \mathbb{E} \prod_{i=1}^{L} \varphi(X_i s), \tag{3}$$

where φ is the Laplace-Stieltjes transform (LST) of $\mathcal{L}(W)$.
If W satisfies (2), then it is natural to refer to $\mathcal{L}(W)$ as the *fixed point of the (supercritical) branching random walk (the BRW) smoothing transform*

$$\mathbb{T} : \mathcal{P}^+ \to \mathcal{P}^+ \cup \{\delta_\infty\};\ \mathcal{L}(U_1) \mapsto \mathcal{L}\left(\sum_{i=1}^{L} X_i U_i\right),$$

where given $Z, U_1, U_2, \ldots$ are conditionally independent identically distributed rvs.

The name is explained as follows. First, we have $\mathbb{T}\mathcal{L}(W) = \mathcal{L}(W)$ (fixed point). Secondly, the martingale $W^{(n)}(\gamma)$, with an appropriate γ, either (a) converges in mean to the rv W having unit mean, or (b) $\lim_{n\to\infty} W^{(n)}(\gamma) = 0$ almost surely; in this case, under some assumptions, there exists a (Seneta-Heyde) norming $\{c_n\}$ which means that $\lim_{n\to\infty} W^{(n)}(\gamma)/c_n = W$ in distribution (properties of the BRW). The dichotomy (a-b) regarding the limiting behaviour of $W^{(n)}(\gamma)$ is justified by Lyons' (1997) change of measure construction (his formula (2)) together with his formulae (5) and (6). The Seneta-Heyde norming is not investigated here. We mention the works Biggins and

Kyprianou (1996, 1997) and Cohn (1997) where this subject is studied for the supercritical BRW with

$$L < \infty \text{ almost surely (a.s.).} \tag{4}$$

Of special interest, as indicated by the title of this paper, is the case that

$$\mathbb{P}\{L = \infty\} > 0, \tag{5}$$

which, for the most part, we will concentrate on. However, as particular cases, we obtain previously known results proved under assumption (4).

As should be clear from the title of the paper, we must introduce a new notion of an *elementary* and a *nonelementary* fixed point. Given $\alpha \in (0, 1]$, we will say that a distribution μ_α is an *α-elementary fixed point* of $\mathbb{T}$ if its LST φ_α satisfies

$$\lim_{s \to +0} \frac{1 - \varphi_\alpha(s)}{s^\alpha} = m, \tag{6}$$

for some finite $m > 0$. Note that a fixed point is 1-elementary if and only if it has finite mean. The set of *elementary fixed points* consists of all α-elementary fixed points, $\alpha \in (0, 1]$. A fixed point will be called *nonelementary* if there is no $\alpha \in (0, 1]$ for which it is α-elementary. Below we provide a rather full description of the elementary fixed points. As the analysis of nonelementary fixed points uses quite different arguments, results in that direction will appear elsewhere.

The paper is organized as follows. In Section 2 we formulate our main results with proofs deferred to Section 3. In Section 4 we lift from our general results the solution to a so-called Pitman-Yor problem. The Pitman-Yor problem is closely related to the problem of the existence of fixed points of the so-called shot noise transforms. These are obtained by putting in (2) $X_i := h(\tau_i)$,where h is a nonnegative Borel measurable function and τ_i, $i = 1, 2, \ldots$ is a Poisson flow. Some comments and references are given in Section 5 and the paper closes with an Appendix where some needed technical results are collected.

In addition to the notation introduced above, other frequently used notation includes:

(a) $\overline{\mu}$ is the *size-biased distribution* corresponding to a given distribution μ with a finite mean; it is defined by the equality

$$\overline{\mu}(dx) := \left(\int_0^\infty y \mu(dy) \right)^{-1} x \mu(dx);$$

if Z is an rv with $\mathcal{L}(Z) = \mu$ then $\overline{Z}$ is an rv with $\mathcal{L}(\overline{Z}) = \overline{\mu}$;
(b) given a σ-finite measure M and $\gamma > 0$, the measure M^*_γ is defined by $M^*_\gamma(dx) := x^\gamma M(dx)$, it is convenient to put $M^* := M^*_1$;
(c) by a *perpetuity* is meant an rv $B_1 + \sum_{i=2}^\infty A_1 \dots A_{i-1} B_i$, where (A_i, B_i), $i = 1, 2, \dots$ are independent copies of a random pair (A, B);
(d) given the BRW smoothing transform $\mathbb{T}$ and $\gamma \in (0,1)$, the *modified transform* $\mathbb{T}_\gamma$ is defined in the same way as $\mathbb{T}$ with the only difference being that the underlying point process has points $\{X_i^\gamma\}_{i=1}^L$. Thus

$$\mathbb{T}_\gamma : \mathcal{P}^+ \to \mathcal{P}^+ \cup \{\delta_\infty\};\ \mathcal{L}(U_1) \mapsto \mathcal{L}\left(\sum_{i=1}^L X_i^\gamma U_i\right).$$

For the reader's convenience, we would like to point out two conventions to be in force throughout the paper.
(**C1**) Clearly, δ_0 always satisfies (2). Hence in what follows we will seek for other fixed-point distributions, not indicating this explicitly.
(**C2**) We will assume that the intensity measure χ satisfies the equality $\chi^*\{0\} = 0$.

Requiring (C1) is only a matter of convenience. (C2) is only needed in the proof of Proposition 1.

2 Results

2.1 The existence and uniqueness

Our first statement gives necessary conditions for the existence of *arbitrary* fixed points of $\mathbb{T}$ and asserts the regular variation of $1 - \varphi(s)$ at zero, where φ is the LST of a fixed point. These results deal with both cases (4) and (5) and were *partially* known under (4) and some additional moment restrictions. See Liu (1998, Theorems 1.1 and 1.2) for details. Note, however, that the validity of Proposition 1 does not require a priori assumptions and hence the result is relatively new even if (4) holds.

Proposition 1. If there exists a fixed point with the LST φ, then
a) there exist at most two values $\beta_1 \le \beta_2$, β_1, $\beta_2 \in (0,1]$ and at least one of these (take $\beta_1 = \beta_2$ in the next equality) such that **Condition** $\mathbf{D}_{\beta_1}$, defined in the line below,

$$\mathbb{E} \sum_{i=1}^L X_i^{\beta_k} = 1,\ k = 1, 2$$

holds and

$$\liminf_{n\to\infty} S_n^{(\beta_1)} = -\infty \text{ almost surely,}$$

where $S_n^{(\beta_1)}$, $n = 0, 1, \dots$ is the random walk:

$$S_0^{(\beta_1)} := 0, \quad S_n^{(\beta_1)} := \sum_{j=1}^{n} \log B_j^{(\beta_1)}, \quad n = 1, 2 \dots$$

and $B_1^{(\beta_1)}, B_2^{(\beta_1)} \dots$ are independent copies of an rv $B^{(\beta_1)}$ with $\mathcal{L}(B^{(\beta_1)}) = \chi^*_{\beta_1}$;

b) $\displaystyle\lim_{s\to+0} \frac{1-\varphi(sz)}{1-\varphi(s)} = z^{\beta_1}$, $z \geq 0$.

In what follows we are considering elementary fixed points. From the above Proposition, we know that the random walk $S_n^{(\beta_1)}$, $n = 0, 1, \dots$ is oscillating or drifting to $-\infty$ (non-oscillating). One of the results of the next Proposition below is that the elementary fixed points correspond to the non-oscillating random walks $S_n^{(\beta_1)}$, $n = 0, 1, \dots$ On the other hand, one may conjecture that nonelementary fixed points could correspond to the random walks of both types.

Theorem 2 contains the necessary and sufficient conditions for the existence of elementary fixed points. Lyons (1997) proved this assertion for fixed points with finite mean (which are 1-elementary fixed points in our terminology) under the side condition that $\mathbb{E}\sum_{i=1}^{L} X_i \log X_i$ is finite.

When Condition D_{β_1} holds, notice that $\mathcal{L}(B^{(\beta_1)}) = \chi^*_{\beta_1}$ and set $R_{\beta_1} := \log B^{(\beta_1)}$. For a distribution σ, set

$$I_{R_{\beta_1}}(\sigma) := \int_{(1,\infty)} \frac{\log x}{\int_0^{\log x} \mathbb{P}\{R_{\beta_1} \leq -y\} dy} \sigma(dx).$$

Theorem 2. For $\alpha \in (0, 1]$, an α-elementary fixed point exists if and only if $\beta_1 = \alpha$ and Condition D_{β_1} together with one of the next three conditions holds

(a) $-\infty < \mathbb{E}R_{\beta_1} < 0$ and $\mathbb{E}(\sum_{i=1}^{L} X_i^{\beta_1}) \log^+(\sum_{i=1}^{L} X_i^{\beta_1}) < \infty$;

(b) $\mathbb{E}R_{\beta_1} = -\infty$ and $I_{R_{\beta_1}}(\overline{\mathcal{L}(\sum_{i=1}^{L} X_i^{\beta_1})}) < \infty$;

(c) $\mathbb{E}R^+_{\beta_1} = \mathbb{E}R^-_{\beta_1} = +\infty$, $I_{R_{\beta_1}}(\chi^*_{\beta_1}) < \infty$ and $I_{R_{\beta_1}}(\overline{\mathcal{L}(\sum_{i=1}^{L} X_i^{\beta_1})}) < \infty$.

The conditions (a), (b), (c) are equivalent to the two requirements:

(d) $\displaystyle\lim_{n\to\infty} S_n^{(\beta_1)} = -\infty$ almost surely;

(e) $I_{R_{\beta_1}}(\overline{\mathcal{L}(\sum_{i=1}^{L} X_i^{\beta_1})}) < \infty$.

Now we would like to reveal an idea of the proof.
(1) *Case* $\alpha = 1$. Fixed points of $\mathbb{T}$ are scale invariant. Thus it suffices to study fixed points with unit mean. By Lemma 14, a fixed point with unit mean exists if and only if the nonnegative martingale $W^{(n)}(\gamma)$, $n = 1, 2, \ldots$ given by (1) converges in mean to it. Therefore, it follows from Lyons' (1997) change of measure construction that such fixed points are closely connected with *perpetuities*. Once this relation has been realized, to deal with the existence of these fixed points, we can use results on perpetuities from the recent comprehensive treatment of Goldie and Maller (2000). Just in this way, either of the conditions (a)-(c) of Theorem 2 ensures the martingale convergence. Thus the case $\alpha = 1$ of Theorem 2 can be viewed as a generalization of Biggins' (1977) martingale convergence theorem.
(2) *Case* $\alpha \in (0, 1)$. As soon as some results are available for 1-elementary fixed points, the corresponding statements for α-elementary fixed points are easily derived via the stable transformation. See the proof of Theorem 2 (case $\alpha \in (0, 1)$) for the precise statement. Section 5 contains some references to other works dealing with the stable transformation.

In the next Proposition we describe the set $\mathcal{H}$ of all elementary fixed points. In fact, this set consists of the fixed points with finite mean ($\alpha = 1$) and the fixed points ($\alpha \in (0, 1)$) obtained from the fixed points with finite mean for the modified transform $\mathbb{T}_\alpha$ via the stable transformation (7). As far as the uniqueness is concerned, we show that provided $\mathcal{H}$ is nonempty, it coincides with the set of all fixed points.

Proposition 3. Let Condition D_{β_1}, (d) and (e) of Theorem 2 be valid. Set $\alpha := \beta_1$.
(a) If $\alpha = 1$ then, for each $m > 0$, $\mathbb{T}$ has a unique (1-elementary) fixed point with mean m.
(b) If $0 < \alpha < 1$ then, for each $m > 0$ in (6), $\mathbb{T}$ has a unique α-elementary fixed point μ_α given by

$$\mu_\alpha(x, \infty) = \int_0^\infty s_\alpha(xt^{-1/\alpha}, \infty)\mu_1(dx), \quad x > 0, \tag{7}$$

where s_α is the strictly stable positive distribution with the index of stability α, and μ_1 is the fixed point with mean m of the modified transform $\mathbb{T}_\alpha$.
(c) $\mathbb{T}$ on $\mathcal{P}^+$ has no other fixed points than those described in (a) and (b).

2.2 Tail behaviour

In this Section we study the tail behaviour of fixed points with finite mean. First we investigate the existence of moments of order $p > 1$. As a by-product we obtain conditions for the L_p-convergence of the martingale $W^{(n)}(1)$. In case (4) the next Proposition is due to Liu (2000, Theorem 2.3). Although his proof works well for the infinite case too, we give an alternative proof for the "$\Rightarrow$" part of the assertion and, when $p \in (1, 2]$, for the "$\Leftarrow$" part.

Proposition 4. Assume that there exists a fixed point $\mathcal{L}(W) \neq \delta_a$, $a \geq 0$ with finite mean. Then, for each fixed $p > 1$, $\mathbb{E}W^p < \infty$ if and only if

$$\mathbb{E}\left(\sum_{i=1}^{L} X_i\right)^p < \infty \text{ and } \mathbb{E}\sum_{i=1}^{L} X_i^p < 1. \tag{8}$$

The next result is obvious. In fact, it suffices to note that if $\mathbb{E}W^p < \infty$ then $W^{(n)}(1) = \mathbb{E}(\liminf_{m\to\infty} W^{(m)}(1)|\mathcal{F}_n)$ and use Jensen's inequality to see that $W^{(n)}(1)$ is bounded in L_p.

Corollary 5. For each fixed $p > 1$, the martingale $W^{(n)}(1)$ is L_p-convergent if and only if (8) and Condition D_1 hold.

Proposition 4 will be proved by using the modern technique based on an appropriate change of measure. However, a simpler proof of the "$\Leftarrow$" part is available as soon as one realizes that under (8) the BRW smoothing transform $\mathbb{T}$ is a strict contraction on some metric space. The reader may want to consult Rösler (1992) and Rachev and Rüschendorf (1995) where the Contraction Principle is used to study transforms more general than ours.

For fixed $\delta > 1$ and $m > 0$, let us consider the set $\mathcal{P}^+(\delta, m)$ of distributions defined as follows

$$\mathcal{P}^+(\delta, m) := \{\mu \in \mathcal{P}^+ : \int_0^\infty x\mu(dx) = m, \int_0^\infty x^\delta \mu(dx) < \infty\}.$$

Given $\mathcal{L}(Y) \in \mathcal{P}^+(\delta, m)$ and a point process whose points $\{X_i\}_{i=1}^L$ satisfy $\mathbb{E}(\sum_{i=1}^L X_i) = 1$ and $\mathbb{E}\left(\sum_{i=1}^L X_i\right)^\delta < \infty$, we have $\mathcal{L}(\sum_{i=1}^L X_i Y_i) \in \mathcal{P}^+(\delta, m)$, where $Y_1, Y_2, \ldots$ are, conditionally on $\{X_i\}_{i=1}^L$, independent copies of Y. Indeed, it is easily seen that

$$\mathbb{E}(\sum_{i=1}^{L} X_i Y_i) = \mathbb{E}(\sum_{i=1}^{L} X_i)\mathbb{E}Y = m.$$

Also by the convexity of the function $x \to x^\delta$, we have

$$\mathbb{E}(\sum_{i=1}^L X_i Y_i)^\delta = \mathbb{E}(\mathbb{E}(\sum_{i=1}^L X_i Y_i)^\delta / \mathcal{F}_1)) \leq$$

$$\leq \mathbb{E}(\mathbb{E}((\sum_{i=1}^L X_i)^{\delta-1}(\sum_{i=1}^L X_i Y_i^\delta))/\mathcal{F}_1)) =$$

$$= \mathbb{E}(\sum_{i=1}^L X_i)^\delta \mathbb{E} Y^\delta < \infty.$$

Thus $\mathbb{T}$ maps $\mathcal{P}^+(\delta, m)$ into itself.
For $\mu_1, \mu_2 \in \mathcal{P}^+(\delta, m)$, let us define the function

$$r_\delta(\mu_1, \mu_2) := \int_0^\infty s^{-\delta-1} \left| \int_0^\infty \exp(isx)\mu_1(dx) - \int_0^\infty \exp(isx)\mu_2(dx) \right| ds.$$

From Lemma 3.1 of Baringhaus and Grübel (1997) we know that provided $\delta \in (1,2)$ r_δ is a metric on $\mathcal{P}^+(\delta, m)$ and that $(\mathcal{P}^+(\delta, m), r_\delta)$ is a complete metric space.

Proposition 6. Let (8) with $p \in (1,2)$ and Condition D_1 be in force. Then the BRW smoothing transform $\mathbb{T}$, on $(\mathcal{P}^+(p, m), r_p)$, is a strict contraction. In particular, $\mathbb{E} W^p < \infty$.

The next assertion refines Theorem 2.2 of Liu (2000) and extends its result to the transforms satisfying (5). First, we show that the power-like tail behaviour does not depend on the type of $\mathcal{L}(\log B^{(1)})$. This features the fixed points under consideration among general perpetuities (see Grincevičius (1975), Theorem 2). Secondly, we indicate the explicit form of the constant in the limit relation (9).

Proposition 7. Assume that for some $b > 1$, $\mathbb{E}(\sum_{i=1}^L X_i) = \mathbb{E}(\sum_{i=1}^L X_i^b) = 1$, $\mathbb{E}(\sum_{i=1}^L X_i^b \log^+ X_i) < \infty$ and $\mathbb{E}(\sum_{i=1}^L X_i)^b < \infty$. Then there exist a fixed point $\mu = \mathcal{L}(W)$ having finite mean, and a positive constant C_b such that

$$\lim_{x\to\infty} x^b \mu(x, \infty) = C_b. \tag{9}$$

Furthermore,
1) if $\mathcal{L}(\log B^{(1)})$ is nonarithmetic then

$$C_b = (\mathbb{E}(\sum_{i=1}^L X_i^b \log X_i))^{-1} \int_0^\infty y^{b-1}(\mu(y, \infty) - N(y, \infty))dy;$$

where N is the σ-finite measure defined by $N^* := \mathcal{L}(B^{(1)}\overline{W})$.
2) if $\mathcal{L}(\log B^{(1)})$ is arithmetic with the span ς then

$$C_b = (\mathbb{E}(\sum_{i=1}^{L} X_i^b \log X_i))^{-1} \sum_{k=-\infty}^{\infty} e^{-\varsigma k} \int_0^{\exp \varsigma k} y^b(\mu(y,\infty) - N(y))dy.$$

3 Proofs of the results

Proof of Proposition 1. We consider the case (5), as the other one, when (4) holds, can be treated similarly. Certainly, we will comment on all points which require different arguments for these cases.
Put $\psi(s) := \dfrac{1-\varphi(s)}{s}$. From (3) we deduce that

$$1 = \lim_{s\to+0} \mathbb{E} \sum_{i=1}^{L} X_i \frac{\psi(X_i s)}{\psi(s)} \prod_{k=1}^{i-1} \varphi(X_k s), \tag{10}$$

the empty product is always taken to be equal to 1. Further we will use arguments given in the proof of Lemma 3.3 in Iksanov and Jurek (2002).
Since

$$0 < \frac{\psi(sz)}{\psi(s)} \le 1 \ \text{ for all } \ z \ge 1, \tag{11}$$

by the selection principle, for any positive sequence s_n which tends to 0 as $n \to \infty$, there exists a subsequence s_{m_n} such that, for $t_n := s_{m_n}$ and $z > 1$, $\dfrac{\psi(t_n z)}{\psi(t_n)}$ converges to some finite limit $\Lambda(z)$ as $n \to \infty$. On the other hand, since each $\psi(t_n z)$ for $n = 1, 2, \ldots$, is a completely monotone function in $z \in (0,\infty)$, and this property is preserved under the limits, $\Lambda(z)$ is also completely monotone, and thus, in particular, it is continuous on $(0,\infty)$. Furthermore,

$$\lim_{n\to\infty} \frac{\psi(t_n z)}{\psi(t_n)} = \Lambda(z) \text{ locally uniformly on } (0,\infty). \tag{12}$$

Also, for fixed $v > 0$, we have that

$$\lim_{n\to\infty} \frac{\psi(t_n v z)}{\psi(t_n v)} = \frac{\Lambda(vz)}{\Lambda(v)} \ \text{ locally uniformly in } z \in (0,\infty). \tag{13}$$

If we would know that $\Lambda(\infty) = 0$, the convergence in (13) was uniform outside 0, and we merely interchanged the limit and the expectation. Under the current circumstances, in view of (10) and Fatou's lemma, we obtain

$$1 = \lim_{n\to\infty} \mathbb{E} \sum_{i=1}^{L} X_i \frac{\psi(X_i t_n v)}{\psi(t_n v)} \prod_{k=1}^{i-1} \varphi(X_k s) \geq \mathbb{E} \sum_{i=1}^{L} X_i \frac{\Lambda(X_i v)}{\Lambda(v)} = \tag{14}$$

$$= \int_0^\infty \frac{\Lambda(vz)}{\Lambda(v)} \chi^*(dz) =: q \in (0,1],\ v > 0.$$

After rewriting this in a more convenient form we get

$$\int_0^\infty \Lambda(vz)(q^{-1}\chi^*(dz)) = \Lambda(v), v > 0.$$

Changing of variable $z := e^{-u}$ gives the integrated Cauchy functional equation (in $\Lambda(e^{-\nu})$). It is known (see, for example, Theorem 8.1.6 in Ramachandran and Lau (1991)) that the solutions to such an equation are of the form

$$\Lambda(v) = p_1(v)v^{\beta_1 - 1} + p_2(v)v^{\beta_2 - 1}, \quad \text{for almost all } v > 0; \tag{15}$$

$$p_k(v) = p_k(vw) \geq 0 \ \text{ for all } \ w \in supp(\chi), k = 1, 2, \tag{16}$$

where $\beta_1 \leq \beta_2$ are determined by the equation

$$q = \int_0^\infty z^{\beta_k} \chi(dz) = \mathbb{E} \sum_{i=1}^{L} X_i^{\beta_k},\ k = 1, 2. \tag{17}$$

In our case, in view of continuity of Λ, (15) holds for all $v > 0$. [Note also that nonzero functions p_k may be different from the identical one only if

$$supp(\chi) = \{\exp(n\gamma) : n \in \mathbb{Z}\},$$

where γ is the largest value that permits such a representation.]

Now we must have $\beta_1 \leq 1$, as otherwise Λ given by (15) is nondecreasing for some $\nu > 0$. By the same reasoning, if $\beta_2 > 1$ then $p_2(v) \equiv 0$. Note further that β_2 cannot be negative, as $v\Lambda(v)$ being the nonincreasing function for small enough ν, would be the limit of *nondecreasing* functions $\dfrac{1 - \varphi(t_n v)}{1 - \varphi(t_n)}$.

Finally, the case $\beta_i = 0$, $i = 1, 2$ is excluded by (17). Indeed, if either (5) or both (4) and $\mathbb{E}L = \infty$ holds then the integral in (17) is infinite. On the other hand, if $\mathbb{E}L < \infty$ then (17) implies that $\mathbb{E}L \leq 1$ which is impossible by Lemma 10(b) below. All in all, it remains to consider two cases which we will study separately: (A1) $0 < \beta_1 = \beta_2 \leq 1$ (which simply means that there is a unique $0 < \beta \leq 1$ satisfying (17); note however that there may be a $\beta > 1$ satisfying (17)) and (A2) $0 < \beta_1 < \beta_2 \leq 1$.

(A1) $\Lambda(v) = p(v)v^{\beta_1 - 1}$, where $p(v) \geq 0$ satisfies (16). We again may repeat the part of the proof of the Lemma 3.3 in Iksanov and Jurek (2002) to conclude that $p(v) \equiv 1$ or equivalently $\Lambda(v) = v^{\beta_1 - 1}$.
It is worth recording here as we need to use twice these arguments. Let us introduce $k(v) := p(-\log v)$. Then $k(v) = v^{1-\beta_1}\Lambda(v)$ is differentiable on $\mathbb{R}$ and

$$k^{'}(v) = v^{1-\beta_1}((1-\beta_1)v^{-1}\Lambda(v) + \Lambda^{'}(v)).$$

Because of the differentiability and periodicity of $k(v)$ there exists a $v_0 > 0$ such that $k^{'}(v_0) = 0$. In fact, $k^{'}(u^n v_0) = 0$, for $u \in supp(\chi)$ and $n = 1, 2, \ldots$ On the other hand, both functions $v^{-1}\Lambda(v)$ and $-\Lambda^{'}(v)$ are positive, nonincreasing and convex. Consequently, for $0 \leq \beta_1 \leq 1$, the equation

$$(1-\beta_1)v^{-1}\Lambda(v) = -\Lambda^{'}(v)$$

either holds identically or has at most two solutions (graphs of the left- and the right-hand side may either coincide or intersect at most at two points). However, the latter means that $k^{'}(v) = 0$ at most at two points, which contradicts the fact that $k^{'}(u^n v_0) = 0$ for $n = 1, 2, \ldots$ Thus, $(1-\beta_1)v^{-1}\Lambda(v) \equiv -\Lambda^{'}(v)$ which implies that $k(v)$ is a constant. Since by (12), $\Lambda(1) = 1$, we conclude $k(v) = 1$, for $v \geq 0$, or equivalently $\Lambda(v) = v^{\beta_1 - 1}$.

With this Λ, the convergence in (13) is uniform outside 0. In view of (C2) we have $\chi^*\{0\} = 0$. Therefore, we may interchange the limit and the expectation in (10) to obtain:

$$\mathbb{E}\sum_{i=1}^{L} X_i^{\beta_1} = 1.$$

Furthermore, appealing to (12) we get

$$\lim_{n\to\infty} \frac{1-\varphi(t_n v)}{1-\varphi(t_n)} = v\Lambda(v) = v^{\beta_1}, \quad \text{for all} \quad v \geq 0.$$

However, the same argument below (11) can be repeated for any subsequence, therefore we conclude that

$$\lim_{s\to+0}\frac{1-\varphi(sz)}{1-\varphi(s)} = z^{\beta_1}, \text{ for all } z \geq 0.$$

Thus in case (A1) the proof of b) and the first part of a) is complete. To check the remaining part of a), set $\Phi_{\beta_1}(s) := e^{\beta_1 s}(1-\varphi(e^{-s}))$ and $\Psi_{\beta_1}(s) := e^{\beta_1 s}\mathbb{E}\{\prod_{i=1}^{L}\varphi(e^{-s}X_i) - 1 + \sum_{i=1}^{L}(1-\varphi(e^{-s}X_i))\}$. We have the (infinite) analogue of the renewal equation given of Lemma 2.3 by Durrett and Liggett (1983)

$$\Phi_{\beta_1}(t) = \mathbb{E}\Phi_{\beta_1}(t - S_n^{(\beta_1)}) - \sum_{i=0}^{n-1}\mathbb{E}\Psi_{\beta_1}(t - S_i^{(\beta_1)}). \tag{18}$$

In view of Lemma 10(a) (see Appendix), the random walk $S_n^{(\beta_1)}, n = 0, 1, \ldots$ is nondegenerate at 0. So let us assume that $\limsup_{n\to\infty} S_n^{(\beta_1)} = +\infty$ a.s. Thus there exists a nonrandom subsequence $\{n_k\}$ which approaches infinity together with k, and such that $\lim_{k\to\infty} S_{n_k}^{(\beta_1)} = +\infty$ a.s. As Φ_{β_1} is bounded on the neighborhoods of infinity and $\lim_{s\to-\infty}\Phi_{\beta_1}(s) = 0$, we have $\lim_{k\to\infty}\mathbb{E}\Phi_{\beta_1}(S_{n_k}^{(\beta_1)}) = 0$. Therefore, $\lim_{k\to\infty}\sum_{i=0}^{n_k}\mathbb{E}\Psi_{\beta_1}(S_i^{(\beta_1)})$ exists and it is nonpositive and possibly infinite. This contradicts the fact that $\Phi_{\beta_1}(0) = 1 - \varphi(1) > 0$. [Note that Biggins (1977, Lemma 3) used similar arguments]. This concludes the proof of the Proposition in case (A1).

(A2) In this case there exists $1 < w \in supp(\chi)$. From (15) and (16) we have, for any $n \in \mathbb{N}$,

$$\Lambda(w^n) = p_1(w^n)(w^n)^{\beta_1-1} + p_2(w^n)(w^n)^{\beta_2-1} = p_1(1)(w^n)^{\beta_1-1} + p_2(1)(w^n)^{\beta_2-1}.$$

If $\beta_2 < 1$, this yields $\lim_{\nu\to\infty}\Lambda(\nu) = 0$, in view of the monotonicity. Consequently, in the same way as in the study of case (A1) we have in (17) $q = 1$, thus proving that Condition D_{β_1} holds. If $\beta_2 = 1$, we just divide the sum in (14) into two parts: $\lim\mathbb{E}\sum_1^L = \lim\mathbb{E}\sum_1^{Z[0,d]} + \lim\mathbb{E}\sum_{Z[0,d]+1}^{L}$, for some $d > 0$. Now we may interchange the limit and either of sums separately. While for the first sum we use the local uniformity of convergence in (13), for the second one we use nonincreasingness of ψ, the dominated convergence and the fact that χ^* is the finite measure. This implies that $q = 1$ and hence Condition D_{β_1} holds.

Now the distribution of the rv $B^{(\beta_1)}$ is well-defined and, moreover, $\mathbb{E}(B^{(\beta_1)})^{\beta_2-\beta_1-\varepsilon} < 1$, for $\varepsilon \in (0, \beta_2 - \beta_1)$. By making use of Jensen's inequality, we conclude that $\mathbb{E}\log B^{(\beta_1)} < 0$ which implies $\lim_{n\to\infty} S_n^{(\beta_1)} = -\infty$ a.s.

We now turn to the proof of the regular variation. By (17), there exists a $1 > w \in supp(\chi)$. Thus (15) and (16) give for any $n \in \mathbb{N}$,

$$(w^n)^{1-\beta_1}\Lambda(w^n) = p_1(w^n) + p_2(w^n)(w^n)^{\beta_2-\beta_1} = p_1(1) + p_2(1)(w^n)^{\beta_2-\beta_1}.$$

Consequently, we have $\lim_{n\to\infty}(w^n)^{1-\beta_1}\Lambda(w^n) = p_1(1)$. Using the monotonicity of Λ, we get

$$wp_1(1) \le \liminf_{\nu\to+0} \nu^{1-\beta_1}\Lambda(\nu) \le \limsup_{\nu\to+0} \nu^{1-\beta_1}\Lambda(\nu) \le \frac{1}{w}p_1(1). \tag{19}$$

In view of (18), $\Phi_{\beta_1}(t - S_n^{(\beta_1)}) - \sum_{i=0}^{n-1}\Psi_{\beta_1}(t - S_i^{(\beta_1)})$, $n = 0, 1, \ldots$ is a martingale. Note that here the functions Φ_{β_1} and Ψ_{β_1} are constructed in the same way as above (18), but with φ and $\{X_i\}$ that we are currently studying. As the random walk $S_n^{(\beta_1)}$ drifts to $-\infty$, the stopping time $\tau = \min\{n > 0 : S_n^{(\beta_1)} < 0\}$ is a.s. finite. By the martingale stopping theorem, we have

$$\Phi_{\beta_1}(t) = \mathbb{E}\Phi_{\beta_1}(t - S_{\tau\wedge n}^{(\beta_1)}) - \sum_{i=0}^{\tau\wedge n-1}\mathbb{E}\Psi_{\beta_1}(t - S_i^{(\beta_1)}) \le \mathbb{E}\Phi_{\beta_1}(t - S_{\tau\wedge n}^{(\beta_1)}).$$

To get the latter inequality, we have used nonnegativity of Ψ_{β_1}(see Durrett and Liggett (1983, Lemma 2.4b) for the proof). Put $\Lambda_{\beta_1}(\nu) := \nu^{1-\beta_1}\Lambda(\nu)$. Using the same t_n as in (12) and the local uniform convergence there gives

$$1 \le \lim_{m\to\infty}\int_{0^+}^1 z^{1-\beta_1}\frac{\psi(t_m v z)}{\psi(t_m v)}\mathcal{L}(e^{S_{\tau\wedge n}^{(\beta_1)}})(dz) =$$

$$= \int_0^1 \frac{\Lambda_{\beta_1}(\nu z)}{\Lambda_{\beta_1}(\nu)}\mathcal{L}(e^{S_{\tau\wedge n}^{(\beta_1)}})(dz) < \infty.$$

The integrand is bounded according to (19). Hence the Lebesgue bounded convergence allows us to pass to the limit as $n \to \infty$, to get

$$1 \le \int_0^1 \frac{\Lambda_{\beta_1}(\nu z)}{\Lambda_{\beta_1}(\nu)}\mathcal{L}(e^{S_\tau^{(\beta_1)}})(dz) =: Q < \infty.$$

Now we may repeat the discussion of the first part of the proof to conclude

$$\begin{aligned}
\Lambda_{\beta_1}(\nu) &= P(v)v^{b_1-1} \text{ for all } v>0;\\
P(v) &= P(vw) \geq 0 \text{ for all } w \in supp(\mathcal{L}(e^{S_\tau^{(\beta_1)}})).
\end{aligned}$$

where b_1, which is necessarily unique as $0 < e^{S_\tau^{(\beta_1)}} < 1$ a.s., is determined by the equation

$$Q = \int_0^1 z^{b_1-1}\mathcal{L}(e^{S_\tau^{(\beta_1)}})(dz).$$

Suppose that $b_1 \neq 1$. Then $b_1 < 1$ which implies that $\Lambda_{\beta_1}(\nu)$ is unbounded near zero. A contradiction. Thus, $\Lambda(\nu) = P(v)v^{\beta_1-1}$, and it remains to copy the proof of case (A1) to show that $P(v) \equiv 1$, $\nu \geq 0$. Repeating all arguments above for each subsequence (like t_n) finishes the proof of the Proposition.

Proof of Theorem 2. *Case* $\alpha = 1$. We first notice that if $\mathcal{L}(W)$ satisfies (2), so is $\mathcal{L}(cW)$, for any $c > 0$. Thus it suffices to study the situation when $\mathbb{E}W = 1$. That the conditions (a)-(c) and (d)-(e) are equivalent follows from Theorem 2.1 and Corollary 4.1 of Goldie and Maller (2000).

The part (a) was studied by Lyons (1997). As to (b), Lyons noticed that his approach works well in this case too, but did not provide arguments. Let us show simultaneously that each of the couples of conditions (d), D_1 and (c), D_1 is sufficient.

Lyons (1997) constructed a probability space $((t,X,\xi),\ \mathcal{F}^*,\ \widehat{\mu}^*)$, where (t,X,ξ) is a space of infinite labelled trees (t,X) with distinguished rays ξ, $\mathcal{F}^* = \cup\mathcal{F}_n^*$, where $\mathcal{F}_n^*$, $n = 1,2,\ldots$ are the σ-fields containing all information about the first n generations in (t,X,ξ), and $\widehat{\mu}^*$ is a probability measure whose "double" restriction $\widehat{\mu}_n$, first to (t,X) then to $\mathcal{F}_n$ satisfies

$$\frac{d\widehat{\mu}_n}{d\mu_n} = W^{(n)}(1), \text{ for all } n \text{ and all } (t,X),$$

where μ_n is the restriction of μ to $\mathcal{F}_n$. Let S be an rv whose distribution is given as follows

$$d\mathcal{L}(S) = (\sum_{i=1}^{L} X_i)d\mathcal{L}(Z).$$

Then with $\mathcal{G}$ being the σ-field generated by the copies of S we have $V_n := \mathbb{E}_{\widehat{\mu}^*}(W^{(n)}(1)/\mathcal{G}) =: V_{1,n} - V_{2,n} \leq V_{1,n}$, where $V_{1,n} = N_1 + \sum_{k=1}^{n-1} M_1 \ldots M_k N_{k+1}$

and $V_{2,n} = \sum_{k=1}^{n-1} M_1 \dots M_k$. Here $M_1, M_2, \dots$ are $\widehat{\mu}^*$ iidrvs with the distribution χ^* which are also $\widehat{\mu}^*$ independent of $\widehat{\mu}^*$ iidrvs $N_1, N_2, \dots$ with the distribution $\overline{\mathcal{L}(\sum_{i=1}^L X_i)}$. Thus, in view of Fatou's lemma and Lemma 11, for the existence of $\mathcal{L}(W)$ it suffices that $V_{1,n}$ be $\widehat{\mu}^*$ a.s. convergent (with a limit being a perpetuity). While the condition (b) ensures the convergence of $V_{1,n}$ by Corollary 4.1(c) of Goldie and Maller (2000), (c) ensures this by Corollary 4.1(d) of the same reference.

Let us now assume that $\mathcal{L}(W)$ exists. The necessity of Condition D_1 follows from the equality $\mathbb{E}W = (\mathbb{E}W_1)\mathbb{E}\sum_{i=1}^L X_i$. Therefore, in the sequel we may and do assume that the distributions χ^* and $\overline{\mathcal{L}(\sum_{i=1}^L X_i)}$ are well-defined. Using formula (7) of Lyons (1997)

$$W^{(n+1)}(1) \geq Y_{n+1},\, n = 0, 1, \dots, \tag{20}$$

where $Y_{n+1} \stackrel{d}{=} M_1 \dots M_n N_{n+1}$ and our Lemma 5.2 with $\varepsilon = 0$ (which is just the formula (6) of Lyons (1997)) gives

$$\limsup_{k\to\infty} M_1 \dots M_k N_{k+1} < \infty, \widehat{\mu}^* \text{a.s.}$$

Suppose that the condition $\lim\limits_{n\to\infty} S_n^{(1)} = -\infty$ $\widehat{\mu}^*$ a.s., fails to hold. This implies $\limsup\limits_{k\to\infty} M_1 \dots M_k = \limsup\limits_{k\to\infty} \exp S_k^{(1)} = +\infty$ $\widehat{\mu}^*$ a.s. A contradiction. Thus, in what follows we assume that $\lim\limits_{k\to\infty} M_1 \dots M_k = 0$ $\widehat{\mu}^*$ a.s.

As $M_1 1_{\{M_1\in[0,1]\}} \dots M_k 1_{\{M_k\in[0,1]\}} \leq M_1 \dots M_k \stackrel{\widehat{\mu}^* a.s.}{\to} 0$, $k \to \infty$, by Lemma 5.6 of Goldie and Maller (2000) we have $I_{-\log^- M_1}(\overline{\mathcal{L}(\sum_{i=1}^L X_i)}) < \infty$. The chain of equalities

$$\infty > I_{-\ln^- M_1}(\overline{\mathcal{L}(\sum_{i=1}^L X_i)}) = \int_{(1,\infty)} \frac{\log x}{\int_0^{\ln x} \widehat{\mu}^*\{-\log^- M_1 \leq -y\}dy} d\widehat{\mu}^*\{N_1 \leq x\} =$$

$$= \int_{(1,\infty)} \frac{\log x}{\int_0^{\log x} \widehat{\mu}^*\{\log M_1 \leq -y\}dy} d\widehat{\mu}^*\{N_1 \leq x\} = I_{R_1}(\overline{\mathcal{L}(\sum_{i=1}^L X_i)}).$$

finishes the proof.

Case $\alpha \in (0, 1)$. Assume that Condition D_{β_1} as well as (d) and (e) of Theorem 2 hold. Put $\alpha := \beta_1$. According to what we have already proved, the modified

transform $\mathbb{T}_\alpha$ (defined in the Introduction) has a fixed point μ_1 with mean m, say. Its LST φ_1 satisfies the equality

$$\varphi_1(s) = \mathbb{E}\prod_{i=1}^{L} \varphi_1(X_i^\alpha s).$$

Set $\varphi_\alpha(s) := \varphi_1(s^\alpha)$. The so defined function is the LST of a distribution μ_α given by (7). Moreover, $\varphi_\alpha(s)$ satisfies (3) and (6). Note that it is easy to check that in that case the right hand side of (3) is well-defined. Thus μ_α is the α-elementary fixed point.

In the reverse direction, let μ_α be an α-elementary fixed point with the LST φ_α. By Proposition 1, Condition D_{β_1} holds with $\beta_1 = \alpha$. It remains to show that the conditions (d) and (e) are necessary. Define the nonnegative nondecreasing and continuous function $\psi_\alpha(s) := \varphi_\alpha(s^{1/\alpha})$ (it is reasonable to call the move from φ_α to ψ_α as *the inverse stable transformation*). It satisfies the equality

$$\psi_\alpha(s) = \mathbb{E}\prod_{i=1}^{\infty} \psi_\alpha(X_i^\alpha s),$$

which is the analogue of (3) for the modified transform $\mathbb{T}_\alpha$, and

$$\lim_{s\to+0} \frac{1-\psi_\alpha(s)}{s} = m. \tag{21}$$

As $\mathbb{T}_\alpha$ verifies Condition D_1, by Lemma 12 ψ_α is the LST of a fixed point of $\mathbb{T}_\alpha$. This fixed point has finite mean in view of (21). Thus, according to the first part of the proof, the conditions (d) and (e) are indeed necessary. The proof is complete.

Proof of Proposition 3. (a-b) Keeping in mind the proof of Theorem 2, it remains to show that given $m > 0$ in (6), there exists a unique α-regular fixed point. The proof below is standard, but it is included here for completeness. Suppose that there exist two α-elementary fixed points whose LST $\varphi_{1,\alpha}$ and $\varphi_{2,\alpha}$(say) satisfy (6) with the same m. From (3), we deduce that the function $\Xi_\alpha(s) := \dfrac{|\varphi_{1,\alpha}(s) - \varphi_{2,\alpha}(s)|}{s^\alpha}$ satisfies the inequality $\Xi(s) \le \mathbb{E}\Xi(B^{(\alpha)}s)$, $s > 0$. Iterating this n times gives $\Xi(s) \le \mathbb{E}\Xi(\exp(S_n^{(\alpha)})s)$, $s > 0$. By Theorem 2, $\exp(S_n^{(\alpha)})$ a.s. goes to zero, when $n \to \infty$. By Lemma 13, $\varphi_{1,\alpha} \equiv \varphi_{2,\alpha}$.

(c) Assume that there exists a nonelementary fixed point with the LST φ^*. By Proposition 1, $1-\varphi^*$ is regularly varying at zero with the index α. In view

of the assumption, the corresponding slowly varying function l, say, is not equivalent to a constant. In fact, we have $l(s) \to +\infty$ when $s \to +0$. If $\alpha = 1$, it is obvious, if $\alpha \in (0,1)$, use the inverse stable transformation (defined in the proof of Theorem 2) and Lemma 12 to reduce this case to the previous one. Now we use the idea of the proof of Theorem 7.4 of Liu (1998). According to parts (a) and (b) of the Proposition, for each $m > 0$ in (6) there exists an α-elementary fixed point with the LST $\varphi_\alpha^{(m)}$, say. Moreover, there exists an $s_m > 0$ such that $1 - \varphi_\alpha^{(m)}(s) \le 1 - \varphi^*(s)$, for all $0 < s < s_m$. Now Lemma 7.3 of Liu (1998) applies. This yields for each $m > 0$, $1 - \varphi_\alpha^{(m)}(s) \le 1 - \varphi^*(s)$, for all $s > 0$. As $\varphi_\alpha^{(m)}(s) \to 0$ as $m \to +\infty$, we get $\varphi^*(s) = 0$, $s > 0$. A contradiction. The proof is complete.

Proof of Proposition 4. We will use the notation exploited in the proof of Theorem 2. Set also $\widehat{W} := \liminf_{n\to\infty} W^{(n)}(1)$. Assume that $\mathbb{E}W^p = \mathbb{E}_\mu(\widehat{W})^p < \infty$. Then $\lim_{n\to\infty} \mathbb{E}_{\widehat{\mu}} \left(W^{(n)}(1)\right)^{p-1} = \mathbb{E}_{\widehat{\mu}}(\widehat{W})^{p-1} < \infty$, the inequality being implied by Lemma 11. An appeal to (20) reveals that the condition (8) is necessary. Assume now that (8) holds or which is equivalent $\mathbb{E}M_1^{p-1} < 1$ and $\mathbb{E}N_1^{p-1} < \infty$. These inequalities ensure that $V_1 := \liminf_{n\to\infty} V_{1,n} < \infty$ (by the general theory of perpetuities, see Goldie and Maller (2000)) and $\mathbb{E}_{\widehat{\mu}^*} V_1^{p-1} < \infty$ (by the triangle inequality in L_{p-1}). By Fatou's lemma, we have

$$\mathbb{E}_{\widehat{\mu}^*}(V_1)^{p-1} \ge \mathbb{E}_{\widehat{\mu}^*} \left(\mathbb{E}_{\widehat{\mu}^*}(\widehat{W}/\mathcal{G})\right)^{p-1}.$$

As it was announced we only consider the case $p \in (1,2]$. In view of Jensen's inequality, the right hand side is bounded below by $\mathbb{E}_{\widehat{\mu}^*}\left(\mathbb{E}_{\widehat{\mu}^*}(\widehat{W}^{p-1}/\mathcal{G})\right) = \mathbb{E}_{\widehat{\mu}^*}\widehat{W}^{p-1}$. It remains to use Lemma 11. The proof is complete.

Proof of Proposition 6. For ν_1 and $\nu_2 \in \mathcal{P}^+(\delta, m)$ with characteristic functions $\psi_1(s)$ and $\psi_2(s)$ respectively, let us denote by $\varphi_i(s)$ the characteristic functions of $\mathbb{T}\nu_i$, $i = 1,2$. For any complex z_i, Z_i with $|z_i| \le 1$ and $|Z_i| \le 1$ we have

$$\left|\prod z_i - \prod Z_i\right| \le \sum |z_i - Z_i|, \tag{22}$$

when the right hand side is finite. Thus we obtain

$$\begin{aligned} |\varphi_1(s)-\varphi_2(s)| &= \left|\mathbb{E}\prod_{i=1}^{\infty}\psi_1(X_is)-\mathbb{E}\prod_{i=1}^{\infty}\psi_2(X_is)\right| \le \\ &\le \mathbb{E}\sum_{i=1}^{\infty}|\psi_1(X_is)-\psi_2(X_is)|\,. \end{aligned}$$

Recalling that the rv $B^{(1)}$ is defined in Proposition 1 and setting $f(s) := \dfrac{|\psi_1(s)-\psi_2(s)|}{s}$, we get

$$r_p(\mathbb{T}\nu_1,\mathbb{T}\nu_2) = \int_0^\infty s^{-p-1}\,|\varphi_1(s)-\varphi_2(s)|\,ds \le$$

$$\le \int_0^\infty s^{-p}\mathbb{E}\sum_{i=1}^{\infty}X_if(X_is)ds = \int_0^\infty s^{-p}\mathbb{E}f(B^{(1)}s)ds \le$$

$$\le \mathbb{E}(B^{(1)})^p\int_0^\infty z^{-p-1}\,|\psi_1(z)-\psi_2(z)|\,dz =$$

$$= \mathbb{E}\left(\sum_{i=1}^{L}X_i\right)^p r_p(\nu_1,\nu_2)$$

Among others this justifies using (22). The proof is complete.

Proof of Proposition 7. Define the function $t(y) := \mathbb{E}(\sum_{i=1}^{L}X_i^y)$. It is convex where it is finite. By convexity, the condition $t(1) = t(b) = 1$ implies $t(y) < 1$ for $y \in (1,b)$. Using Jensen's inequality gives $\mathbb{E}\log B^{(1)} = \mathbb{E}(\sum_{i=1}^{L}X_i\log X_i) < 0$. On the other hand, $\mathbb{E}(\sum_{i=1}^{L}X_i)^b < \infty$ implies $\mathbb{E}(\sum_{i=1}^{L}X_i)\log^+(\sum_{i=1}^{L}X_i) < \infty$. Therefore, by Theorem 2 with $\alpha = 1$ there exists a fixed point $\mu = \mathcal{L}(W)$ having finite mean $m > 0$, say.

The proof goes almost the same path as that of Proposition 1.2 in Iksanov and Kim (2003), where the tail bahaviour of fixed points of the (shifted Poisson) shot noise transforms has been studied. Keeping this in mind, we only give a sketch of the proof and refer the interested reader to Iksanov and Kim (2003) for details. There is a point of essential difference between this work and the just cited one. In place of the simple perpetuity (24) investigated in Iksanov and Kim (2003) we should use the other perpetuity (23) found by Liu (2000, Lemma 4.1). At this point we would like to stress that these perpetuities

are quite different and (23) is not reduced to (24) even for the shot noise transforms.

We start with the renewal equation

$$P_b(x) = \int_{-\infty}^{\infty} P_b(x-y)\rho_b(dy) + Q_b(x),$$

where

$$P_b(x) := e^{-x} \int_0^{\exp x} y^b \mu(y,\infty)dy,$$

$$Q_b(x) := e^{-x} \int_0^{\exp x} y^b(\mu(y,\infty) - N(y,\infty))dy$$

and $\rho_b(dy) := e^{(b-1)y}\mathcal{L}(B^{(1)})(de^y)$. We have the equality of distributions which essentially shows that $\overline{W}$ with $\mathcal{L}(\overline{W}) = \overline{\mu}$ is the perpetuity

$$\overline{W} \stackrel{d}{=} B^{(1)}\overline{W} + C, \tag{23}$$

where $\overline{W}$ is independent of $(B^{(1)}, C)$ and $\mathbb{E}f(C) = \mathbb{E}(\sum_{i=1}^L X_i f(\sum_{\substack{1\le k\le L\\ k\neq i}} X_k))$, for any nonnegative Borel functions f.

Set $I(x) := \int_0^\infty y^x(\mu(y,\infty) - N(y,\infty))dy$. For $\beta < b-1$, we have $\mathbb{E}(B^{(1)})^\beta = \mathbb{E}(\sum_{i=1}^L X_i^{\beta+1}) < 1$ and, by Proposition 4 $\mathbb{E}W^{\beta+1} < \infty$ which imply

$$0 < I(\beta) = (\beta+1)^{-1}\mathbb{E}W^{\beta+1}(1 - \mathbb{E}(B^{(1)})^\beta) < \infty.$$

Applying the c_β-inequality to (23) results in

$$m^{-1}\mathbb{E}W^{\beta+1}(1 - \mathbb{E}(B^{(1)})^\beta) \le (2^{\beta-1} \vee 1)M_\beta,$$

$$m^{-1}\mathbb{E}W^{\beta+1}(1 - \mathbb{E}(B^{(1)})^\beta) \ge (2^{\beta-1} \wedge 1)K_\beta,$$

where the constant

$$M_\beta := \mathbb{E}(\sum_{i=1}^L X_i)^{\beta+1} \text{ , if } \beta \in (0,1] \quad \text{and} \quad := \mathbb{E}W^\beta\mathbb{E}(\sum_{i=1}^L X_i)^{\beta+1}, \text{ otherwise,}$$

is identified in Lemma 4.2 of Liu (2000), and the constant

$$K_\beta := \mathbb{E}(\sum_{i=1}^L X_i)^{\beta+1} \text{ , if } \beta \ge 1 \quad \text{and} \quad := \mathbb{E}W^\beta\mathbb{E}(\sum_{i=1}^L X_i)^{\beta+1}, \text{ otherwise,}$$

can be obtained in the similar way. Letting β go to $b-1$ along some subsequence gives

$$0 < mb^{-1}(2^{b-2} \wedge 1)K_{b-1} \leq I(b-1) \leq mb^{-1}(2^{b-2} \vee 1)M_{b-1} < \infty,$$

which according to Lemma 9.2 of Goldie (1991) implies that $Q_b(x)$ is directly Riemann integrable.

By the key renewal theorem for the whole line we obtain:
1) if $\mathcal{L}(\log B^{(1)})$ is nonarithmetic then

$$\lim_{x\to\infty} P_b(x) = (\mathbb{E}(\sum_{i=1}^{L} X_i^b \log X_i))^{-1} \int_0^\infty y^{b-1}(\mu(y,\infty) - N(y,\infty))dy := C_b;$$

2) if $\mathcal{L}(\log B^{(1)})$ is arithmetic with the span ς then for all $x \in \mathbb{R}$

$$\lim_{n\to\infty} P_b(x+\varsigma n) = (\mathbb{E}(\sum_{i=1}^{L} X_i^b \log X_i))^{-1} \sum_{k=-\infty}^{\infty} Q_b(x+\varsigma k) := C_b(x).$$

Thus we only need to consider the arithmetic case.
From the results of Grincevičius (1975) it follows that there exist $d_1 > 0$ and $d_2 < \infty$ such that, for all x large enough, we have

$$0 < d_1 \leq x^b \mu(x,\infty) \leq d_2 < \infty.$$

Taking this into account, we can prove that $G_b(x)$ slowly varies at ∞. The working from Iksanov and Kim (2003) may be repeated, but we must have an alternative proof of the fact that $\lim_{n\to\infty} \dfrac{N(t_n u,\infty)}{\mu(t_n u,\infty)} \in [0,1]$ provided the limit exists for some sequence $t_n \to \infty$, as $n \to \infty$. This follows easily from (23) which implies that $\overline{\mu}(x,\infty) \geq N^*(x,\infty)$, thus giving $\mu(x,\infty) \geq N(x,\infty)$ (use the LST). It remains to show that

$$\lim_{n\to\infty} e^{(x+\varsigma n)}\mu(e^{x+\varsigma n},\infty) = C_b(0) \text{ locally uniformly in } e^x \text{ on } (0,\infty),$$

which together with slow variation will give the result.

4 A Pitman-Yor problem as a particular case

In Pitman and Yor (2000, p.35) the following problem concerning distributions on the nonnegative half line has been mentioned. For what distributions

ν there exists a distribution μ with finite mean such that

$$\overline{Y} \overset{d}{=} A\overline{Y} + Y, \tag{24}$$

where $\mathcal{L}(A) = \nu$, $\mathcal{L}(Y) = \mu$ and $\mathcal{L}(\overline{Y}) = \overline{\mu}$.

To point out the solution to the above problem, let us introduce the random walk $T_0 := 0$, $T_n := \sum_{k=1}^{n} \log A_k$, $k = 1, 2, \ldots$, where A_1, $A_2, \ldots$ are independent copies of the rv A. We now intend to explain how the next result originally obtained in Iksanov and Kim (2003) follows from Theorem 2 (case $\alpha = 1$).

Proposition 8. The condition

$$\lim_{n\to\infty} T_n = -\infty \text{ almost surely} \tag{25}$$

is necessary and sufficient for the existence of a distribution $\mu \neq \delta_0$ satisfying (24). Given $m > 0$, there exists a unique μ with mean m.

Recall that a right-continuous and nonincreasing function $g : (0, \infty) \to [0, \infty)$ allows us to define the generalized inverse function $g^{\leftarrow}$ as follows

$$g^{\leftarrow}(z) := \inf\{u : g(u) < z\}, \text{ if } z < g(0^+) \ ; \ := 0, \text{ otherwise.}$$

We do not know how the problem could be solved on using only the equality (24). Hence, the alternative representation of the rv Y is given next.

Lemma 9. (Iksanov and Kim (2003)) a) Assume that a distribution of the rv Y with finite mean $m > 0$ satisfies (24), where the rv A is such that $\gamma := \mathbb{P}\{A = 0\} \in [0, 1)$. Then we have

$$Y \overset{d}{=} m\gamma + \sum_{i=1}^{\infty} Y_i h(\tau_i), \tag{26}$$

where $Y_1, Y_2, \ldots$ are independent copies of Y, which are also independent of a Poisson flow $\{\tau_i\}, i \geq 1$ with intensity 1; $h : (0, \infty) \to [0, \infty)$ is right-continuous and nonincreasing function defined by

$$h^{\leftarrow}(x) = \int_x^{\infty} z^{-1} \mathcal{L}(A)(dz), x > 0.$$

Therefore, we have $\int_0^{\infty} h(z)dz = 1 - \gamma$.
b) If a distribution of Y satisfies (26), where the intensity of the Poisson

flow is equal to $\lambda > 0$, and a right-continuous and nonincreasing function $h : (0,\infty) \to [0,\infty)$ satisfies the equality

$$\lambda \int_0^\infty h(z)dz = 1 - \gamma,$$

Then $\mathcal{L}(Y)$ solves (24), where $\mathcal{L}(A)$ is defined by $\mathcal{L}(A)(dx) = -\lambda x h^{\leftarrow}(dx)$.

We are ready to check that the assertion of Proposition 8 with $\mathbb{P}\{A = 0\} = 0$ is contained in Theorem 2. To this end, we use the observation that $\mathcal{L}(Y)$ solving (24) is a fixed point of the Poisson shot noise transform (see Iksanov and Yurek (2002) for more details) or, in other words, $\mathcal{L}(Y)$ satisfies (26) and vice versa. Hence, let us set in Theorem 2 $X_i = h(\tau_i)$, where h and τ_i are defined in Lemma 9.

Condition D_1 is equivalent to $\int_0^\infty h(u)du = 1$ (when applying part b) of Lemma 9 we can always take a Poisson flow with unit intensity). Thus, all that we need is to show that the condition (d) of Theorem 2 implies the condition (e) of the same Theorem. Under the current notation, this reduces to showing that (25) implies

$$I_{\log A}(\overline{\mathcal{L}(\sum_{i=1}^\infty h(\tau_i))}) < \infty. \tag{27}$$

To see this, it suffices to note that $\chi(dt) = -h^{\leftarrow}(dt)$ and $\mathcal{L}(A) = \chi^*$.
Condider two cases: (1) $-\infty < \mathbb{E}\log A < 0$ and (2) $\mathbb{E}\log A = -\infty$ or $\mathbb{E}\log A$ does not exist.
Case 1. Our task simplifies to checking that

$$\mathbb{E}\log A \in (-\infty, 0) \ \text{ implies } \ \mathbb{E}\log(1 + \overline{\sum_{i=1}^\infty h(\tau_i)}) < \infty. \tag{28}$$

We have $\mathbb{E}\log A \in (-\infty, 0)$ implies $\mathbb{E}\log(1 + A) < \infty$ and Condition D_1 ($\mathbb{E}\sum_{i=1}^\infty h(\tau_i) = 1$) implies $\mathbb{E}\log(1 + \sum_{i=1}^\infty h(\tau_i)) < \infty$. It is easy to observe that $\overline{\sum_{i=1}^\infty h(\tau_i))} \overset{d}{=} \sum_{i=1}^\infty h(\tau_i) + A$ (use LST's). Therefore, applying the inequality (29) gives (28).
Case 2. Set

$$g(x) := \frac{x}{\int_0^x \mathbb{P}\{\log A \le -y\}dy}, \ f(x) := xg(\log(1 + x)), x \ge 0.$$

$$t(x) := (x \wedge 1)(1 + g(\ln(1 + x))), x \ge 0.$$

Since $g(x)/x$ is nonincreasing, we have $g(x+y) \le g(x) + g(y)$, for all $x, y \ge 0$ (subadditivity). This together with nondecreasingness of $g(x)$ and the inequality

$$\log(1+x+y) \le \log(1+x) + \log(1+y), \tag{29}$$

results in

$$f(x+y)/(x+y) \le f(x)/x + f(y)/y \ \ \text{, for all} \ \ x, y \ge 0.$$

Consequently, the function $t(x)$ is submultiplicative. By Theorem 25.3 of Sato (1999) the integrability of a submultiplicative function with respect to an infinitely divisible distribution is equivalent to the integrability (near infinity) of the function in question with respect to the corresponding Lévy measure. As $(-1)h^{\leftarrow}(dx)$ defines the Lévy measure of the infinitely divisible distribution $\mathcal{L}(\sum_{i=1}^{\infty} h(\tau_i))$, we have

$$\int_0^{\infty} t(x)\mathcal{L}(\sum_{i=1}^{\infty} h(\tau_i))(dx) < \infty \ \ \text{if and only if} \int_1^{\infty} t(x)(-1)h^{\leftarrow}(dx) < \infty.$$

By the same criterion, Condition D_1 implies that both integrals $\int_1^{\infty} x h^{\leftarrow}(dx)$ and $\int_0^{\infty} x\mathcal{L}(\sum_{i=1}^{\infty} h(\tau_i))(dx)$ converge. This and the above display together implies

$$\infty > \int_1^{\infty} f(x)\mathcal{L}(\sum_{i=1}^{\infty} h(\tau_i))(dx) \ \ \text{if and only if} \ \ \infty > \int_1^{\infty} f(x)(-1)h^{\leftarrow}(dx) = .$$

$$= \int_1^{\infty} g(\log(1+x))d\mathbb{P}\{A \le x\} \ \ \text{if and only if} \ \ \infty > \int_0^{\infty} g(\log^+ x)d\mathbb{P}\{A \le x\}$$

It is clear that the former inequality is equivalent to (27). Recall that we consider the case when $\log A$ has no finite mean. Thus, by (1.19) of Kesten and Maller (1996), the latter inequality is equivalent to (25), and the asserted follows.

5 Comments and some references

5.1. Fixed points of the smoothing transforms with a.s. finite number of summands have been receiving much attention in the literature. It is worth

mentioning that usually more or less restrictive additional moment assumptions have been imposed. The basic techniques and results for the case of finitely many summands were developed in the basic paper of Durrett and Liggett (1983). Liu (1998) extended their results to the case of finite but random number of summands. Fixed points of the BRW smoothing transforms with almost surely finite number of summands were studied (without using the name) by Biggins (1977), Biggins and Kyprianou (1996, 1997, 2001, 2003), Kyprianou (1998), Liu (2000). Except for the papers on fixed points of the shot noise transforms (see point 5.2 below), we are aware of only four works dealing with fixed points of the smoothing transforms with *infinite* number of summands. These are Lyons (1997), Rösler (1992), Caliebe and Rösler (2003a,b). Note also that the last three papers investigate fixed points concentrated on the whole line. Since the extensive surveys were presented, for example, in Rösler (1992) and Liu (1998), we refrain from listing other works on the subject here.
5.2. Details regarding *some* fixed points of Poisson shot noise transforms can be found in Iksanov (2002a,b), Iksanov and Jurek (2002), Iksanov and Kim (2003, 2004). The Pitman-Yor problem was solved in Iksanov and Kim (2003) by using an approach different from that taken here.
5.3. In the area of smoothing transforms the notion of "stable transformation" is due to Durrett and Liggett (1983). See also Guivarc'h (1990) and Liu (1998) for further development of this concept.
5.4. The trick based on the martingale stopping we have used in the proof of Proposition 1 had come to our attention from Durrett and Liggett (1983, Theorem 2.18). Similar idea, but sometimes in different forms, was much exploited in works connected with convergence of Markov processes and/or martingales. Here we only mention Biggins and Kyprianou (1997, Sections 6 and 8; 2001; 2003, Lemma 1) who developed approaches of such a flavour in relations to fixed points of the smoothing transforms.

6 Appendix

For ease of references we collect here some facts taken mainly from other sources.
Lemma 10. (a) (Liu (1998), Lemma 1.1) If $\mathbb{P}\{X_i = 0 \text{ or } 1, \text{ for all } i \leq L\} = 1$ then there are no fixed points.
(b) (Liu (1998), Lemma 3.1) If fixed points exist then $\mathbb{E}L > 1$.

Lemma 11. Let θ be a finite measure and ν a probability measure on a σ-field $\mathcal{F}$. Suppose that $\mathcal{F}_n$ are increasing sub-σ-fields whose union generates $\mathcal{F}$ and that the restriction of θ to $\mathcal{F}_n$ is absolutely continuous with respect to the restriction of ν to $\mathcal{F}_n$ with Radon-Nikodym derivative Y_n. If $Y := \limsup_{n\to\infty} Y_n < \infty$ then for fixed $\varepsilon \geq 0$ $\int X^{1+\varepsilon} d\nu < \infty$ if and only if $\int X^{\varepsilon} d\theta < \infty$. If one of these is finite then $\int X^{1+\varepsilon} d\nu = \int X^{\varepsilon} d\theta$.
Proof. When $\varepsilon = 0$ this is just Lemma 10.2 from Lyons and Peres (2004) where among others it was shown that $\theta(A) = \int X d\nu$ for all $A \in \mathcal{F}$. Hence, $X^{1+\varepsilon} d\nu = X^{\varepsilon} d\theta$ and the Lemma follows.

The main message of the next Lemma is that the stable and inverse stable transformations give a one-to-one correspondence between fixed points of $\mathbb{T}$ with $\mathbb{E}\sum_{i=1}^{L} X_i^{\beta_1} = 1$, $\beta_1 \in (0,1)$ and those of the modified transform $\mathbb{T}_{\beta_1}$. For the elementary fixed points this observation was exploited in the proof of the $\alpha \in (0,1)$ case of Theorem 2.
Lemma 12. Assume that Condition D_{β_1} holds and a nonnegative nonincreasing and continuous function $f(s), s \geq 0$ satisfies $f(0) = 1$ and

$$f(s) = \mathbb{E}\prod_{i=1}^{L} f(X_i s),$$

and $1 - f(s)$ regularly varies at zero with index β_1.
Then f is the LST of a fixed point of $\mathbb{T}$.
Proof. The proof follows the similar path as that of part (d) of Theorem 7.1 of Liu (1998). A close inspection of Liu's proof reveals that his assumption that ϕ is the LST of a fixed point is not needed. (Also the restrictions $L < \infty$ and (H1) are of no importance for the stated *here* result to hold). It simply suffices to require that ϕ satisfies the conditions of our Lemma.
Indeed, assume first that $\beta_1 \in (0,1)$. First we want to show how one may construct an LST g, say, such that

$$\lim_{s\to+0} \frac{1 - f(s)}{1 - g(s)} = 1.$$

Even though $1 - f$ is regularly varying, it is far from being obvious to us that such functions g do exist. By Theorem 1.7.6 and a variant of Theorem 1.5.8 of Bingham, Goldie and Teugels (1989),

$$h(s) := (\alpha/\Gamma(1-\alpha)) \int_0^s \int_0^\infty e^{-ux}(1 - f(1/x))dxdu$$

is the nonnegative function with completely monotone derivative and $\lim_{s\to+0} \frac{1-f(s)}{h(s)} = 1$. If $\lim_{s\to\infty} h(s) = A \in (1,+\infty]$ then there exists a finite nonzero s_1 such that $h(s_1) = 1$. In that case, the function
$g(s) := 1-h(s_1(1-e^{-s/s_1}))$ is a wanted LST. Indeed, the superposition of two nonnegative functions with completely monotone derivatives is a nonnegative function with completely monotone derivative and $1 - e^{-s} \sim s$ as $s \to 0$. If $\lim_{s\to\infty} h(s) = A \in (0, 1]$ then choose $g(s) := 1 - h(s)$.

Lemma 7.2 of Liu (1998) applies since his condition $\rho(\alpha) \leq 1$ is implied by our Condition D_{β_1}. Hence, the statement of Lemma 7.3 of the same reference is true in our case too. Thus we have that f is completely monotone as the pointwise limit of the sequence of the LSTs

$$\psi_0(s) := g(s), \quad \psi_n(s) := \mathbb{E} \prod_{i=1}^{L} \psi_{n-1}(sX_i), n = 1, 2, \dots$$

Since by assumption $f(0) = 1$, $f(s)$ is the LST of a (nondefective) distribution and therefore it is the LST of a fixed point. Assume now that $\beta = 1$. Unfortunately, it is not clear to us how we could construct a function like h in this case. Thus we will use another argument. For each $\gamma \in (0, 1)$, consider the collection of point processes $Z_\gamma(\cdot)$ with the points $\{X_{\gamma,i}\}_{i=1}^{L}$ such that $X_{\gamma,i} := X_i^{1/\gamma}$, $\overline{i = 1, L}$. Define also $f_\gamma(s) := f(s^\gamma)$. By what we have already proved, for each $\gamma \in (0, 1)$, $f_\gamma(s)$ are the LST's of the BRW smoothing transform constructed by the point process $Z_\gamma(\cdot)$. As $f(s) = \lim_{\gamma\to1} f_\gamma(s)$ and $f(0) = 1$, $f(s)$ is the LST of a fixed point. The proof is complete.

The following Lemma contains an observation implicitly made in Theorem 1 of Athreya (1969). It provides an easy way of showing that when restricted to the set of the α-elementary fixed points, fixed points are unique up to the scale (see the proof of Proposition 3).

Lemma 13. Let $\varphi_{1,\alpha}(s)$ and $\varphi_{2,\alpha}(s)$ be the Laplace-Stieltjes transforms of distributions and assume that they satisfy (6) with the same m. If for large enough positive integer n, the function $\Xi_\alpha(s) := \frac{|\varphi_{1,\alpha}(s) - \varphi_{2,\alpha}(s)|}{s^\alpha}$ satisfies the inequality

$$\Xi(s) \leq \mathbb{E}\Xi(C_n s),\ s > 0,$$

where $\{C_n\}$ is a sequence of rvs that goes to zero almost surely, as $n \to \infty$, then $\varphi_{1,\alpha}(s) \equiv \varphi_{2,\alpha}(s)$.

Lemma 14. A fixed point with unit mean exists if and only if the nonnegative martingale $W^{(n)}(\gamma)$, $n = 1, 2, \dots$ given by (1) converges in mean to it.
Proof. One implication is obvious, hence let us assume that a fixed point $W(\gamma)$ with unit mean exists. Let φ be its LST. Arguing as in the proof of Lemma 12, we have that $\varphi(s) = \lim_{n\to\infty} \varphi_n(s)$, where

$$\varphi_0(s) := e^{-s}, \quad \varphi_n(s) := \mathbb{E}\prod_{i=1}^{L} \varphi_{n-1}(sX_i), n = 1, 2, \dots$$

Since $\varphi_n(s) = \mathbb{E}e^{-sW^{(n)}(\gamma)}$, $W^{(n)}(\gamma)$ weakly converges to $W(\gamma)$ when $n \to \infty$. Thus $W^{(n)}(\gamma)$ cannot converge to zero almost surely. Therefore, it must converge to $W(\gamma)$ in mean (see the last paragraph on page 3). The proof is complete.

Note that the above result (with different proof) is also given in Theorem 2.2(1) of Caliebe and Rösler (2003).

Acknowledgement

My sincere thanks go to Dr Amke Caliebe and Dr Andreas Kyprianou whose detailed comments on the manuscript have led to the essential improvement of presentation.

References

[1] Athreya, K.B., 1969. *On the supercritical one dimensional age dependent branching processes.* Ann.Math.Stat. **40**, 743-763.

[2] Baringhaus, L., Grübel, R., 1997. *On a class of characterization problems for random convex combinations.* Ann.Inst.Stat.Math. **49**, 555-567.

[3] Biggins, J.D., 1977. *Martingale convergence in the branching random walk*, J. Appl. Prob. **14**, 25-37.

[4] Biggins, J.D., Kyprianou, A.E., 1996. *Branching random walk: Seneta-Heyde norming.* In Trees: Proceedings of a Workshop held in Versailles June 14-16, 1995, eds B. Chauvin, S. Cohen, A. Rouault. Birkhauser, Basel, 31-50.

[5] Biggins, J.D., Kyprianou, A.E., 1997. *Seneta-Heyde norming in the branching random walk*. Ann. Prob. **25**, 337-360.

[6] Biggins, J.D., Kyprianou, A.E., 2001. *Measure change in multitype branching*. Preprint, http://www.shef.ac.uk∼st1jdb/mcimb.html.

[7] Biggins, J.D., Kyprianou, A.E., 2003. *Fixed points of the smoothing transform; the boundary case*. Preprint, http://www.shef.ac.uk∼st1jdb/tsttbc.html.

[8] Bingham, N.H., Goldie, C.M., Teugels, J.L., 1989. *Regular variation*. Cambridge University Press, Cambridge.

[9] Caliebe, A., Rösler, U., 2003a *Fixed points with finite variance of a smoothing transformation*. Stochastic Process. Appl. **107**, 105-129.

[10] Caliebe, A., Rösler, U., 2003b *Fixed points of a smoothing transformation with finite expectation: closing a gap*. Preprint, .

[11] Cohn, H., 1997. *Norming constants and convergence of supercritical branching random walks*. Preprint, http://www.ms.unimelb.edu.au∼harry/harryres/norming.ps

[12] Durrett, R., Liggett, T., 1983. *Fixed points of the smoothing transformation*. Z. Wahrscheinlichkeitsth. **64**, 275-301.

[13] Goldie, C.M., 1991. *Implicit renewal theory and tails of solutions of random equations*. Ann. Appl. Prob. **1**, 126-166.

[14] Goldie, C.M., Maller, R.A., 2000. *Stability of perpetuities*. Ann. Prob. **28**, 1195-1218.

[15] Grincevičius, A. K., 1975. *One limit distribution for a random walk on the line*, Lithuanian Math. J. **15**, 580-589.

[16] Guivarc'h, Y., 1990. *Sur une extension de la notion semi-stable*. Ann. Inst. Henri Poincaré. **26**, 261-285.

[17] Iksanov, A.M., 2002a. *On positive distributions of the class L of self-decomposable laws*. Theor.Probability and Math.Statist. **64**, 51-61.

[18] Iksanov, A.M., 2002b *On perpetuities related to the size-biased distributions*. Theory Stoch. Proc. **8** , 128-135.

[19] Iksanov, A.M., Jurek, Z.J., 2002. *On fixed points of Poisson shot noise transforms*. Adv. Appl. Prob. **34**, 798-825.

[20] Iksanov, A.M., Kim, C.S., 2003. *On a Pitman-Yor problem*. Preprint, http://www.unicyb.kiev.ua∼iksanov.

[21] Iksanov, A.M., Kim, C.S., 2004. *New explicit examples of fixed points of Poisson shot noise transforms*. Austr. New Zealand J. Stat., to appear.

[22] Kesten, H., Maller, R.A., 1996. *Two renewal theorems for general random walks tending to infinity*. Probab. Theory Relat. Fields. **106** , 1-38.

[23] Liu, Q., 1998. *Fixed points of a generalized smoothing transformation and applications to the branching random walk*. Adv.Appl.Prob. **30**, 85-112.

[24] Liu, Q., 2000. *On generalized multiplicative cascades*. Stoch. Proc. Appl. **86**, 263-286.

[25] Lyons, R., 1997. *A simple path to Biggins's martingale convergence for branching random walk*. In Athreya, K.B., Jagers, P. (eds.). Classical and Modern Branching Processes, IMA Volumes in Mathematics and its Applications, vol. **84**, Springer, Berlin, pp. 217-221.

[26] Lyons, R., Peres, Y., 2004. *Probability on Trees and Networks*, Cambridge University Press, Cambridge, to appear.

[27] Pitman, J., Yor, M., 2000. *Infinitely divisible laws associated with hyperbolic functions*. Technical Report **581**, Dept. Statistics, U.C. Berkeley.

[28] Rachev, S.T., Rüschendorf, L., 1995. *Probability metrics and recursive algorithms*. Adv. Appl. Prob. **27**, 770-799.

[29] Ramachandran, B., Lau, K.S., 1991. *Functional equations in probability theory*. Academic Press, New York.

[30] Rösler, U., 1992. *A fixed point theorem for distributions*. Stochastic Process. Appl. **42**, 195-214.

[31] Sato, K., 1999. *Lévy processes and infinitely divisible distributions*, Cambridge University Press, Cambridge.